Analemmatic Sundials and Mean Time

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Abstract: *Some developments for classical analemmatic sundials are presented. A first one is Roger Bailey's proposal for "Seasonal Markers". Further we discuss possibilities for analemmatic sundials which register mean time. Common analemmatic sundials register apparent time and not mean time. If we construct "split analemmatic sundials" then direct reading of mean time is possible with quite good approximation. Possibilities for the construction of split analemmatic sundials - some of them suggested in [7] - are shown and calculation formulas are extended to dials with inclined dial plane and/or inclined gnomon.*

Short description of different types of analemmatic sundials

1. "Classical" analemmatic sundials with linear date scale along the North-South-axis.

Analemmatic Sundials consist of a movable gnomon and fixed hour marks along an ellipse. If a person or a gnomon is positioned on the actual date point of the analemmatic sundial, then the shadow line points to local apparent time. The hour marks may also be shifted along the ellipse in that way, that apparent time of the zone meridian is registered instead of local apparent time. In some special cases the hour ellipse can change into a circle or a straight line.

Analemmatic sundials can be regarded as a parallel projection of the universal equatorial ring dial, where the direction of the gnomon is identical with the direction of the projection. In most cases this direction is rectangular to the horizontal dial plane. The shadow casting gnomon must be shifted according to the Sun's declination along the projection of the polar axis of the equatorial ring dial. Usually these projection points are called "date points", according to the Sun's declination on this day. The formulas for calculating such dials can be found in many sundial books (for instance in [6]).

Nowadays often analemmatic sundials are defined in a broader sense. The direction of the projection, which is also the direction of the gnomon, may be of any angle which is not parallel to the dial plane, and the dial plane itself may be inclined too (look at [1]). In this article we restrict all considerations to the fact, that the inclination of the gnomon and the inclination of the orthogonal vector of the dial plane may be inclined but are to be seen within the meridian plane[1]. Therefore the dial plane may be inclined, but must not show any deviation to East or West. We measure the inclination of the dial plane by the position of its normal (orthogonal) vector. In all formulas here the direction of the Plane's normal vector κ and the gnomon's direction γ is measured as the angle distance from the zenith. Additionally we define angles to North as negative and to South as positive. (If the gnomon's angle $\gamma = 0°$, then the gnomon is

With major axis a, minor axis b, D(0/d) as the date point on the minor axis, latitude φ, gnomon inclination γ, plane inclination κ, apparent time T and the hour points $P(x_T / y_T)$ we have the formulas:

$$\overline{OB} = b = a * \frac{\sin(\varphi - \gamma)}{\cos(\gamma - \kappa)}$$

$$\overline{OD} = d = a * \frac{\tan(\delta) * \cos(\varphi - \gamma)}{\cos(\gamma - \kappa)} = b * \frac{\tan(\delta)}{\tan(\varphi - \gamma)}$$

$$x_T = a * \sin(T) \quad and \quad y = b * \cos(T) \tag{1}$$

Positive x-axis is always East and positive y-axis is North.

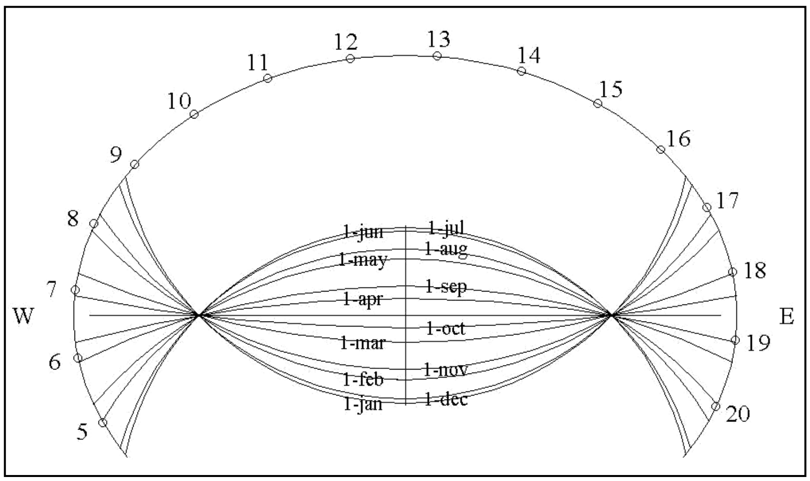
[1] A very good overview about different sundial types as the projection result of the equatorial dial is given by Bruno Ernst in [3].

vertical; a pole pointing gnomon has $\gamma = (\varphi - 90°)$, independent of the dial's inclination. In horizontal planes we have $\kappa = 0°$.)

Finally we must not forget a restriction: Analemmatic sundials cannot be used all the year long, if the dateline becomes about equal or longer than the minor axis of the ellipse of the hour points. Then the shadow cast by the gnomon may be much too short or its direction may show two intersections[2] with the hour ellipse.

Seasonal Markers

R. Rohr described a construction by Lambert to find the times of Sun rise/set[3] in classical analemmatic sundials with horizontal dial plane and vertical gnomon (look at [5]). A circle running through the date point of the day and the focuses of the hour ellipse intersects the ellipse exactly at the time of Sun rise/set. The graphic by Fer de Vries shows such circles. The radius of the different circles is given by $r = a * \cos(\varphi) / \sin(2\delta)$. This construction is not possible for analemmatic sundials with inclined gnomon or inclined dial plane.



Roger Bailey suggested an excellent approximation for finding time and azimuth of Sun rise / set for any particular day (look at [2]). For all days he constructed the intersection points of the major axis (in E-W-direction) and the line from the date point on the N-S-axis to the calculated time point of Sun rise/set for this day. Of course all marks for Sun rise are on the Western part of the major axis and all marks for Sun set symmetric on its Eastern part. For each day of the year there exists one mark for Sun rise. R. Bailey found that all these marks do not differ very much in middle latitudes. Therefore one "Seasonal Rise Mark" for Sun rise (and one for Sun set) somewhere in the middle of all points is a good approximation for all other rise/set marks.

This led to a very simple method for finding Sun rise/set and its azimuth. It was discussed by R. Bailey on a NASS-conference and it is explained in the image below.

With the gnomon's inclination γ or the dials plane inclination κ and the Sun declination δ the position of the Seasonal Markers on the major axis are fixed by:

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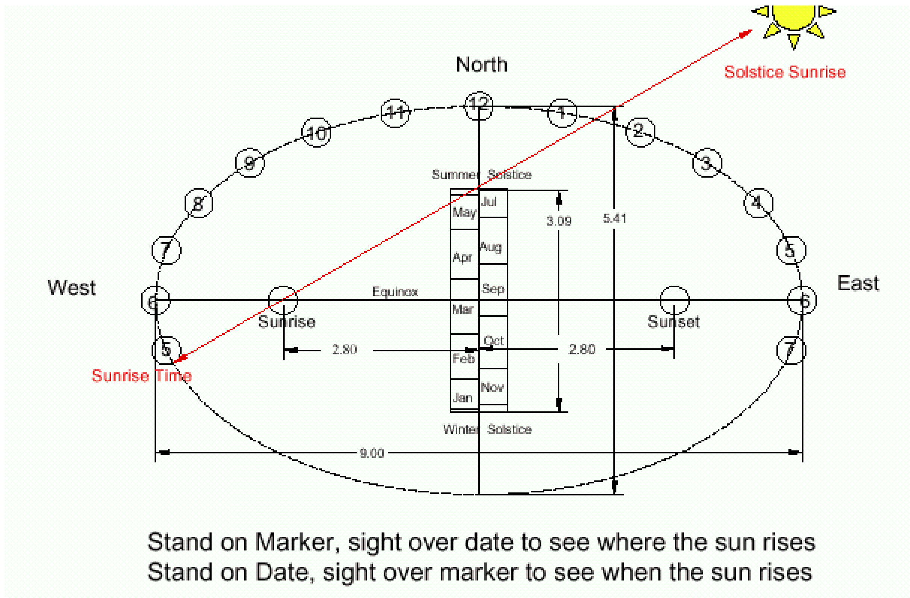
[2] On those days the gnomon's shadow is "going backwards" for a (small) part of the day and then the Sun's azimuth is decreasing in Northern latitudes at this part of the day.

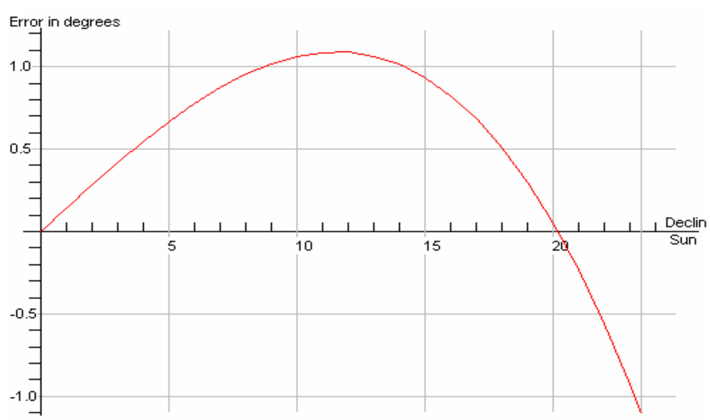
[3] Here Sun rise is defined as the time when the altitude of the Sun's centre is 0°. Influences of atmospheric refraction, sea level, *etc.* are neglected.

$$x = \pm a * \frac{\cos(\varphi - \gamma) * \cos(\varphi) * \cos(\kappa)}{\cos(\kappa - \gamma)} * \sqrt{1 - \tan^2(\delta) * \tan^2(\varphi)}$$



NASS presentation by Roger Bailey [2]

If we select (about!) $\delta = 20{,}2°$ for the fixed "Seasonal Markers", then the absolute error gets minimal. The graph below shows the difference between true and approximated direction of Sun rise/set in degrees for all days with positive Sun declination (for $\delta = 20{,}2°$ and latitude 48°). Negative Sun declination changes the sign of the error.



As one can see, the sight line along the Seasonal Mark never differs more than 1.1° from the true direction. This is a value which can be neglected in analemmatic sundials with common dimensions.

If the gnomon is inclined within the meridional plane, then this method is still possible. In sundials with inclined dial plane all becomes a bit more complicated, even if the gnomon is vertical, *i. e.* its inclination is 0°. This because of the fact, that all angles between the minor axis of the dial and any straight line within the dial plane intersecting the N-S-axis are different from the azimuth, because azimuth angles are measured in the horizontal plane. But with slight modification one also could find azimuth and time of Sun rise/set with help of Seasonal Marks. The error shown above and measured in the horizontal plane is independent of the gnomon's or the dial's inclination. In not horizontal dials with not vertical gnomon (inclination γ <> 0°) Seasonal Markers are not useable at all.

2. Analemmatic sundials with one single 8-slope as date line

The most famous dial of this type is the sundial in Brou. It led many gnomonists to the idea that the 8-slope on the N-S-axis allows reading mean time if the gnomon is positioned on the date point of the 8-slope. This idea of course is not correct in this general way.

But under some specific conditions such a sundial can register exactly mean time at noon. Then the 8-slope is a mirrored Analemma containing Equation of Time[4]. Its date points have the same y-coordinates as the date points of the linear date scale in the "classical" analemmatic sundial and the value of the EoT fixes the x-value of the 8-slope as $x = a * \sin(\varepsilon)$, where $\varepsilon = -EoT$.

If now the gnomon is positioned on the date point of the analemma, then at noon mean time is registered nearly exactly. If we want to read local mean time at noon very exactly, then the formulas for the 8-slope in textbox (2) have to be used.

(2)

If $\varepsilon = -EoT =$ (mean time - apparent time) we get the Analemma point $P(x_\delta, y_\delta)$ by using the symbols and formulas from textbox (1) :

$x_T = a * \sin(\varepsilon)$ and $y_T = b * \cos(\varepsilon)$

$x_\delta = \dfrac{x_T * (b-d)}{(y_T - d)}$ and $y_\delta = d$

But in nearly all cases this is sufficient :

$x_\delta = a * \sin(\varepsilon)$ and $y_\delta = d$

Of course that only corrects for mean time at 12 o'clock (noon) by shifting the gnomon in E-W-direction. Additionally we can say, that such a shift does not influence time reading for time points where the shadow line is exactly in E-W-direction, and therefore no correction at all is done for those hours. That's why a single 8-slope cannot correct for the EoT for all hours of the day.

If the gnomon is put on the 8-slope, then the already mentioned fixed Seasonal Markers for Sun rise/set do not show correct results. Yet Bailey's method and result for the "classical" analemmatic sundial may still be used. This is because the distances of the date point on the 8-slope and of the date point of the linear scale along the minor axis are equal. And if this correlating point on the linear scale along the N-S-axis is used instead of the date point on the 8-slope, then all works as discussed before.

3. Split analemmatic sundials

To understand one reason for split dials let us consider the following: If the Sun is late (in relation to mean time) then correction for EoT at noon can be done by shifting the position of the gnomon eastwards parallel to the major axis according to the value of the EoT. Any shift in the rectangular N-S-direction does not influence time reading at noon as long as the shadow line is exactly from South to North. If we now at the same day wish to do an additional correction for the late Sun at 6 p.m., then the position of the date point can be shifted in direction South without any influence of the first time reading at noon. But then for 6 a.m. the same correction must be done in the opposite direction because consecutive hour points on the ellipse at 6 a.m. are running from S to N, and at 6 p.m. from N to S. Therefore we need two

[4] Here the Equation of Time is defined as: EoT = apparent local time – mean local time

separated 8-slopes, one for times before noon and one for the afternoon. Thus we construct split analemmatic sundials.

In split analemmatic sundials two analemmas (8-slopes) are needed, one for morning and one for afternoon. To avoid confusions two separated 8-slopes should be drawn, and then of course also two separated halves of hour ellipse are necessary. Calculations show, that standing on the date point of such

For a given date with Sun declination δ and Equation of time EoT and selected apparent times t_1 and t_2 the analemma points $P({}_1x_2, {}_1y_2)$ are calculated by: (3)

$$ {}_1x_2 = \frac{\sin(\varepsilon)*\left[\sin(\varphi-\gamma)*\left(\sin(t_1)-\sin(t_2)\right)-M_\delta*\sin(t_1-t_2)\right]}{\sin(\varphi-\gamma)*\sin(t_1-t_2)-M_\delta*\left(\sin(t_1)-\sin(t_2)\right)} $$

$$ {}_1y_2 = \frac{M_\delta*\left[\cos(\varepsilon)*\sin(\varphi-\gamma)*\sin(t_1-t_2)-M_\delta*\left(\sin(T_1)-\sin(T_2)\right)\right]+\sin(\varepsilon)*\sin^2(\varphi-\gamma)*\left[\cos(t_1)-\cos(t_2)\right]}{\cos(\gamma-\kappa)*\left[\sin(\varphi-\gamma)*\sin(t_1-t_2)-M_\delta*\left(\sin(t_1)-\sin(t_2)\right)\right]} $$

where $M_\delta = \tan(\delta)*\cos(\varphi-\gamma)$ and

$t = T - \varepsilon$ with ε as the negative Equation of Time (EoT) for the given date. [But be careful: here EoT is defined as EoT= $-\varepsilon$ = (apparent time – mean time).]

an 8-slope allows reading mean time all through the year with very good approximation. We even can calculate 8-slopes which register exactly mean time (or Standard time) for two particular times in the morning and for two particular times in the afternoon. Of course it is possible to find many different methods to search for the “best-fitting” analemma points. Some of them are discussed now.

Some Methods to calculate split analemmatic sundials

The type of a split analemmatic sundial and its calculation depends on the fact as what we define as the “best fitting” mean time reading. Here we depict four methods.

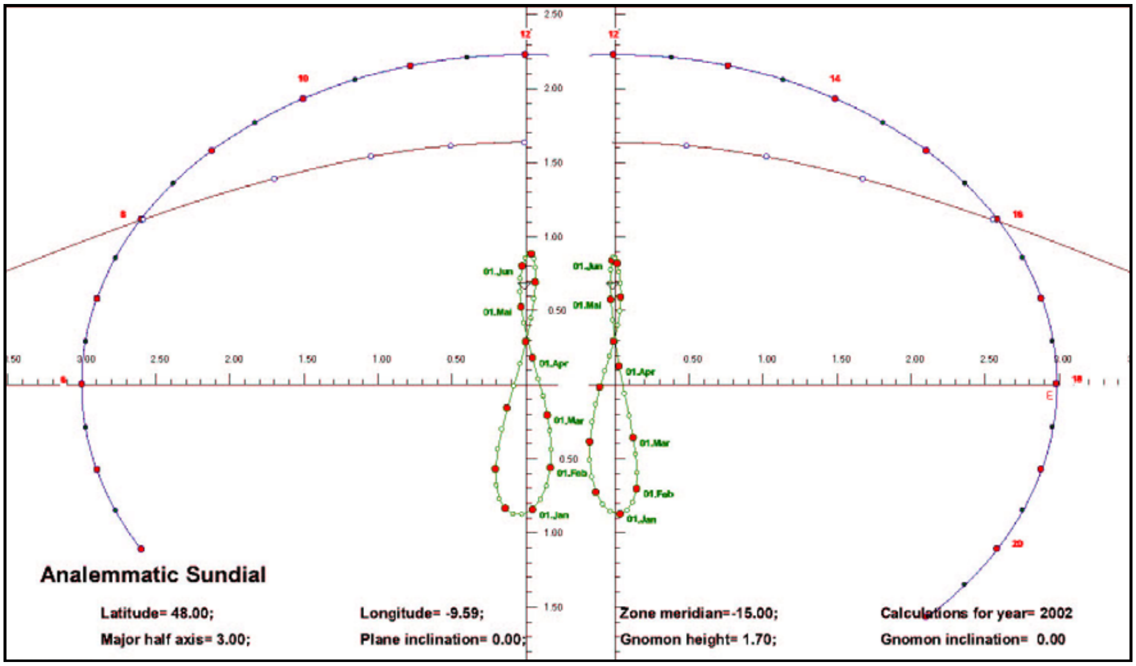
3.1 Method 1 (“exact for 2 selected times”)

Method 1 constructs points on the 8-slope, which allow exact reading of Standard time for 2 selected times a.m. and for 2 time points p.m. all year long. Fred Sawyer suggested this in [7], page 7.11-7.12.: There he suggests to observe the shadow lines of two particular (mean) times and to calculate their intersection point. If that is done for the days of the year, then all these points together build a kind of analemma. If the gnomon is put on the analemma point of this day, then mean time can be read exactly at these times.

If the formulas given by Fred Sawyer are extended to analemmatic sundials with an inclined gnomon or dial plane we get the results of textbox (3).

To judge the quality of such a split sundial, we must calculate the errors of time reading for the rest of the hours for the different days of the year. This can be done with the formulas given in text box (5) at the end of this article. Here we show such a split dial for latitude 48° which registers exactly mean time at 9:00 and 11:00 in the morning and at 13:30 and 17:30 in the afternoon.

The error evaluation for this split sundial shows, that the maximal error for reading mean time between 8 a.m. and 7 p.m.is 77 sec in the morning and 132 sec in the afternoon. The error calculation was done according the formulas of textbox (5).



3.2 <u>Method 2 ("Mean value of method 1")</u>
Method 2 calculates a weighted average of analemma points from method 1: For each pair of N selected time points (morning or afternoon), the points according method 1 are built. Then a weighted average of the coordinates of the calculated points is built. (The type of weighting is discussed in [7]). The formulas for the weighted average are shown in textbox (4).

If we have N selected apparent times t with intersection points P(${}_ix_k$, ${}_iy_k$) (4)
which are calculated according method1 we have to calculate Z_k by

$$\cot(Z_k) = \frac{\sin(\varphi - \gamma) * \cos(t_k) - \cos(\varphi - \gamma) * \tan(\delta)}{\sin(t_k) * \cos(\gamma)}$$

Afterwards we can calculate the average point as discussed in [6]

$$x = \left(\sum_j \sum_{k \neq j} {}_j x_k * |\cos(Z_k)| \right) \Big/ (N-1) \sum_k |\cos(Z_k)|$$

$$y = \left(\sum_j \sum_{k \neq j} {}_j y_k * |\sin(Z_k)| \right) \Big/ (N-1) \sum_k |\sin(Z_k)|$$

3.3 <u>Method 3 ("Minimal average of absolute errors")</u>
Here for each day a range search for the best fitting analemma points for registering mean time is done. The search with refined grids is started with the date point on the single analemma. So for each selected day the analemma point with the smallest weighted average of all absolute errors on this day is calculated. The calculation has to be done for all hours which have been fixed together with their statistical "weight" by the user. Times with Sun below the horizon should be excluded in all calculations, otherwise the values could be fraudulently altered.

This method was discussed first by Kenneth Seidelmann in his mathematical investigation on the Longwood Sundial (look at [8]). When he did the calculations, he first had to save altitude and azimuth of the Sun on a magnetic tape for all hours of a complete year! Afterwards he could start the calculations for the different date points. One can guess, that a rather "huge" computer was needed. Now all calculations can be done easily within a few seconds with any common personal computer and compiled software.

Brian Albinson used a very similar method in additional VB-macros in an Excel-spreadsheet by R. Bailey and H. Sonderegger. But Excel needs quite a long time, because such macros are rather slow. The results differ a little bit from mine for different reasons: different program languages and different search grids are used. Another reason lies in the exactness of the calculations in the different program languages. And the results of course depend on the fact, if times with negative Sun altitudes (*i.e.* Sun below the horizon) are excluded or not.

The error for reading mean time can be calculated with methods discussed in [7]. Slight modifications for an inclining gnomon or dial plane are necessary. Textbox (5) shows the modified results.

We ask for apparent time t when the shadow line of the gnomon with inclination γ (5) and base point $P(x_\delta / y_\delta)$ will pass through time point (x_T, y_T). P is the point for a particular day with Sun declination δ. If we write

$$a = (y_T - y_\delta) * \cos(\gamma - \kappa), \quad b = (x_\delta - x_T) * \sin(\varphi - \gamma), \quad c = (x_\delta - x_T) * \cos(\varphi - \gamma) * \tan(\delta)$$

then time t is calculated with

$$\sin(t) = \frac{ac \mp b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2} \quad \text{and} \quad \cos(t) = \frac{bc \pm a\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}$$

The error itself for reading mean time is found with (t – T) .

3.4 Method 4 ("Minimum of standard deviation")

This method is very similar to method 3. But here the grid search is done for points with minimal daily standard deviation of the absolute error.

Annotation: The author offers a free program which calculates all the discussed types of analemmatic sundials. You are invited to download this software from webpage http://web.utanet.at/sondereh.

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[10] Sundial Mailing List: *Contrubutions from Roger Bailey, Mike Deamicis-Roberts, Fer de Vries, et al.*

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