Mean Time Sundial With A Cone Gnomon

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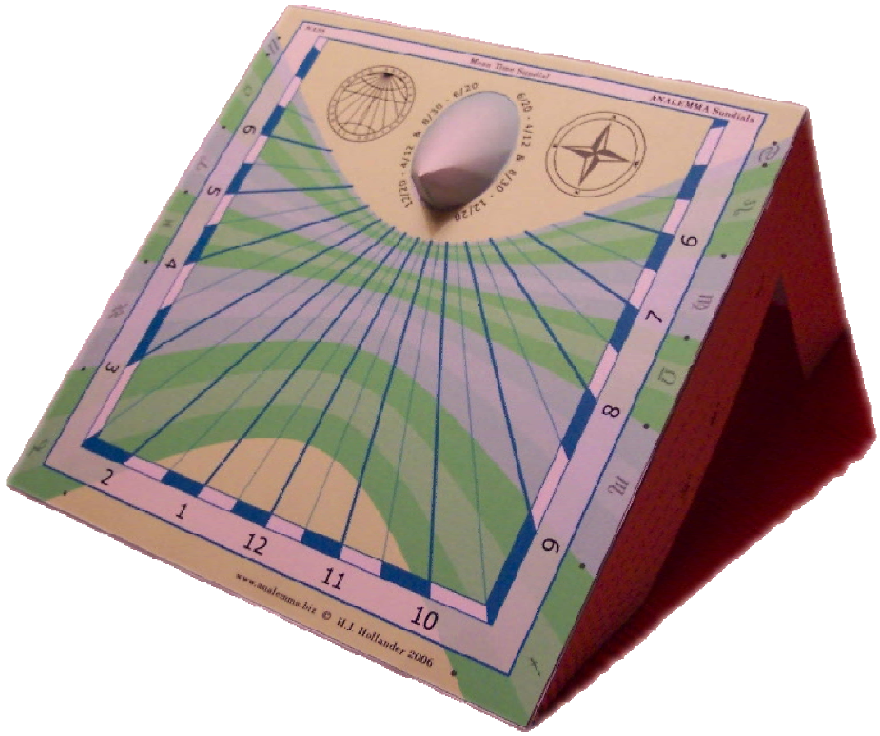
<u>Basic principles of the mean time sundial with a cone gnomon</u>

On the sundial plane the hour lines and date lines are present. These lines look very similar to a regular sundial. Also a shadow casting cone is present. The central axis of the cone is parallel to the earth's axis. One can see the shadow of the left side of the cone and the right side of the cone separately. For which period of the year you have to use the shadow of which side of the cone is marked on the sundial.



Look at the date line of today and look at the shadow of the correct side of the cone. Read the mean time on the date line.

Below you will find the mathematical background of the mean time (or clock time) sundial with a cone gnomon.

<u>The connection between the longitude of the sun, the declination of the sun and date lines on a sundial</u>

As regularly done by dialists we use 30 degree multiples (or even better, 10 degree multiples) of the longitude (λ) of the sun to mark the date. The seasons are marked by $\lambda=0^\circ$ for spring, $\lambda=90^\circ$ for summer, $\lambda=180^\circ$ for autumn and $\lambda=270^\circ$ for winter. Often the lines of the multiples of $\lambda=30^\circ$ are marked with the zodiac signs.

In general each declination of the sun has two λ's associated. One for the period of the winter solstice to the summer solstice (λ_{w-s}) and one for the period of the summer solstice to the winter solstice (λ_{s-w}). For instance the line for spring and autumn is the same line (declination of the sun is 0°) and has $\lambda_{w-s} = 0^\circ$ and $\lambda_{s-w} = 180^\circ$ associated. In general λ_{w-s} and λ_{s-w} are related by:

$$\lambda_{w-s} = 180^\circ - \lambda_{s-w} \qquad with \quad \lambda_{s-w} \in [90^\circ, 270^\circ]$$

notice that λ_{w-s} and λ_{s-w} represent the same declination of the sun and therefore represent one "date line" on a standard sundial with a gnomon. We are now to build a mean time (or clock time) sundial with the same "date lines".

<u>The algorithm of the mean time/clock time sundial</u>

Follow these steps:

1. Choose a valid combination of λ_{w-s} and λ_{s-w}

2. Associate $\lambda_{w\text{-}s}$ with the shadow of one side of the cone and $\lambda_{s\text{-}w}$ with the other side (see addendum for a good suggestion on this choice)
3. Choose a moment in mean time or clock time
4. Calculate the shadow lines for this moment and the λ's of both sides of the cone (formula's are presented below)
5. Calculate the intersection of the 2 shadow lines
6. Mark this point on the sundial plane, this point is part of the hour line of the time chosen in step 3 and is also part of the "date line" associated with $\lambda_{w\text{-}s}$ and $\lambda_{s\text{-}w}$
7. repeat step 1 to 6 many times for different $\lambda_{w\text{-}s}$ and $\lambda_{s\text{-}w}$ and different moments and connect these points to hour lines and date lines

Addendum Step 2.
I developed these algorithms during February 2006. During the very nice discussions I had with Fred Sawyer about this subject, Fred suggested a way to associate the λ's so that the shadow of the cone will always intersect the date lines x. This makes the sundial very user friendly and this concept is incorporated in the algorithm ever since. *[Editor's Note: See the following article.]*

To force this, one has to associate the period that the sun is fast in respect to mean time (or clock time) to the left side of the cone with the sun behind you.

Calculation of the cone
The central axis of the cone should be parallel to the earth axis and therefore, in general, it is oblique to the plane of the sundial. The cone can be drawn on a flat paper, cut out and put together to the 3 dimensional shape.

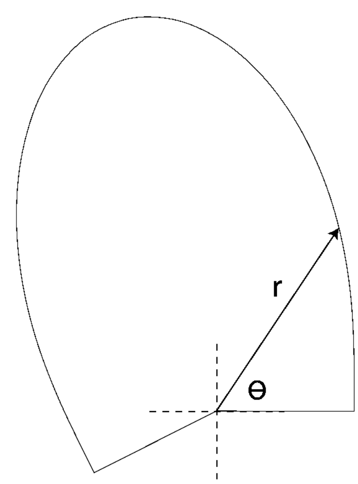
Defining:
h : the length of central axis of the cone from the sundial plane to the apex
γ : the apex half angle
φ : the angle between the cone axis and the sundial plane
(φ = 90 means the cone axis is perpendicular to the sundial plane)
(for horizontal sundials φ can be interpreted as the latitude)

In polar coordinates (r,θ) the cone can be plotted with:

$$r = \frac{h}{\cos\gamma + \sin\gamma \tan(90^\circ - \varphi)\cos\left(\dfrac{\theta}{\sin\gamma}\right)} \qquad \text{using}$$

$\theta \in [0\ ,\ 2\pi \sin\gamma]$.

An example of an unfolded cone is shown here.



To be able to draw the ellipse in the sundial plane where the cone will be mounted we define further:

q : the distance between the heart of the sundial (where the cone axis intersects the sundial plane) and the center point of the ellipse

the ellipse itself: $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = 1$, we notice that the apex of the cone is located at

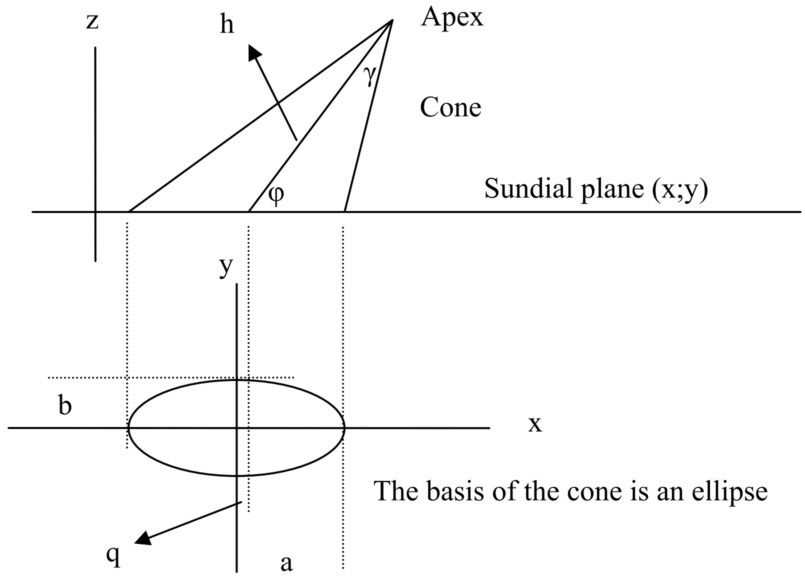
$x_{apex} = h\cos\varphi + q$,

$z_{apex} = h\sin\varphi$ and

$$a = \frac{h\sin\gamma}{2\sin(\varphi-\gamma)} + \frac{h\sin\gamma}{2\sin(\varphi+\gamma)}$$

$$q = a - \frac{h\sin\gamma}{\sin(\varphi+\gamma)}$$

$$b = ha\tan\gamma\sqrt{\frac{1}{a^2-q^2}}$$



The apex of the cone will cast a shadow by a sunbeam at a certain time and declination of the sun. This shadow point of the apex can be calculated with standard methods for sundials with a gnomon.

Defining this shadow point as (P_1, P_2), we notice that this point will be part of the shadow line of the cone. Also the shadow line will be tangent to the ellipse. Let this shadow line be $(y - P_2) = (x - P_1)r$.

it can be shown that $r = \dfrac{-P_1P_2 \pm \sqrt{(a^2P_2^{\,2} + b^2P_1^{\,2} - a^2b^2)}}{(a^2 - P_1^{\,2})}$

It must be checked that (P_1, P_2) is outside the ellipse which can be done with: $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 \geq 1$

One has to distinguish between the shadow of the left and right side of the cone with the sun behind you. This can be done with:

$$\textit{if } \ sign(P_1) = sign(P_2) \Rightarrow \textit{left shadow line is the smaller } |r|$$
$$\textit{if } \ sign(P_1) = -sign(P_2) \Rightarrow \textit{left shadow line is the larger } |r|$$

I want to thank Fred Sawyer (President of the *North American Sundial Society*) for his support and good suggestions on the algorithm (see above) and also Fer de Vries (Secretary of the Dutch Sundial Society *De Zonnewijzerkring*) for his support and checks on the algorithms.

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The 2006 Sawyer Dialing Prize has been awarded to Hendrik Hollander for his innovative design of a mean-time planar sundial with oblique conical gnomon and modified hour lines and day curves – resulting in a sundial adapted to modern timekeeping while retaining the aesthetic appeal of the familiar dial face.